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NON-LINEAR FREE VIBRATION ANALYSIS OF COMPOSITE PLATES WITH INITIAL IMPERFECTIONS AND IN-PLANE LOADING

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Abstract—In this paper the non-linear response of layered composite plates with initial imperfections and subjected to in-plane preload is considered. The basic equations used in the analysis are those corresponding to an accurate shear deformation theory which employs parabolic shear strain variations across the thickness and requires no correction factors. The five governing non-linear equilibrium equations of the problem are reduced to a single non-linear ordinary differential equation (ODE) using a single mode approach in conjunction with the Galerkin method. Numerical results are obtained for two problems: (i) large amplitude vibration of imperfect plates and (ii) small oscillations in the vicinity of a static buckled position of an in-plane loaded two-layered (0/90) plate.

NOTATION

a, b, h	length, width, and total thickness of plate
E_x, E_y	Young's moduli
G_{xy}, G_{xz}, G_{yz}	shear moduli
f_s	static deflection of in-plane loaded plate (see eqn (18))
f_l	static deflection of plate at limit load (see eqn (22))
f_T	total deflection of preloaded plate (see eqn (19))
f_t	time-dependent part of the deflection of preloaded plate (see eqn (19))
h_1, h_2	thicknesses of the individual layers
N_x, N_y, N_{xy}; M_x, M_y, M_{xy}; M'_x, M'_y, M'_{xy}	stress-resultants (see eqn (5))
N_c, N_l	bifurcation buckling and limit loads in static compression
N_x^*, N_y^*	applied in-plane loads in the x- and y-directions on the plate
q_i, q_o	applied transverse loads in the z-direction on the inner and outer surface of the plate ($q = q_i + q_o$)
t	time coordinate
u, v, w	displacement components in the x-, y-, z-directions
u_0, v_0, w_0	displacement components in the x-, y-, z-directions at the mid-plane
u_1, v_1	shear rotations of the normals at the mid-plane
$\bar{w}_0$	initial imperfections
x, y, z	Cartesian coordinate system
$\gamma_{xy}, \gamma_{xz}, \gamma_{yz}$	shear strains
$\varepsilon_x, \varepsilon_y, \varepsilon_z$	normal strains
ν_{xy}	Poisson's ratio
ρ	mass density of plate material
ρ_1, ρ_2	mass densities of individual layers
$\sigma_x, \sigma_y, \sigma_z$	normal stresses
$\tau_{xy}, \tau_{xz}, \tau_{yz}$	shear stresses
Ω_l	linear radian frequency of plate (see eqn (12))
Ω_{NL}	non-linear radian frequency of plate (see eqn (17))
Ω_p	radian frequency of preloaded plate.

INTRODUCTION

Due to their increased use as light-weight high-strength structural components, vibration of fibre-reinforced plates and shells has been the subject of significant current interest, as discussed by Bert (1982) and Leissa (1981). The bulk of the research on laminated plates has been concerned with small amplitude vibrations, static buckling and post-buckling analysis of perfect plates.

Recent studies on the vibration analysis of plates are directed towards the analysis of imperfect plates. Dynamic buckling of isotropic structures when initial imperfections, in-plane inertia and geometric nonlinearities are taken into account (individually) has been treated extensively by Bolotin (1964), and Ambartsumian *et al.* (1966). More recently, combined treatment of this problem for isotropic plates is given by Pasic and Herrmann (1984). Considering large amplitude vibration of an imperfect plate Hui (1984) showed that, unlike the more familiar hardening type nonlinearity of perfect plates, geometric imperfections (depending on their magnitude) may change the characteristics to soft-spring type.

The effects of geometric imperfections on the frequency–load interaction of preloaded (biaxially compressed) composite plates has been studied by Hui and Leissa (1983) for homogeneous plates and by Hui (1985) for angle-ply laminated plates. Elishakoff *et al.* (1987) showed that frequency of a preloaded cylindrical panel vanishes when the load reaches the limit load.

Studies on the effects of initial imperfections and preload on the frequency of buckled structures has been made by Elishakoff *et al.* (1984) and de Souza (1985, 1987) using simple models which represent a number of structures. These models include symmetric, asymmetric, and general non-symmetric cases. Elishakoff *et al.* (1984) have derived the closed form formulae for the natural frequency of these various models. In his studies de Souza (1985, 1987) also investigated the effect of damping on the vibration of preloaded structures.

It can be noted here that in all the above mentioned studies the basic equations used are those corresponding to the classical thin plate theory. It is a well-established fact that the effects of shear deformation are more pronounced in composite plates. Thus, it is more appropriate to treat the vibration of buckled composite plates using a shear deformation theory. In this paper vibration of a preloaded laminated plate has been considered using a more recently proposed parabolic shear theory (PST) (Bhimaraddi and Stevens, 1984; Bhimaraddi, 1987a) in which shear strains are assumed to vary parabolically across the thickness.

In the present analysis the non-linear (von Karman type) equations of Bhimaraddi (1987b) are modified to include the initial imperfections. Since the present theory includes the thin plate theory (TPT) and the Mindlin-type constant shear theory (CST) as special cases, the comparison of these with PST has also been made. A brief description of PST will be presented in what follows.

BASIC EQUATIONS OF THE PARABOLIC SHEAR THEORY

The components of displacements (refer to Notation) are assumed as

$$u(x,y,z,t) = u_0(x,y,t) + \xi u_1(x,y,t) - z\frac{\partial w_0}{\partial x}$$

$$v(x,y,z,t) = v_0(x,y,t) + \xi v_1(x,y,t) - z\frac{\partial w_0}{\partial y}$$

$$w(x,y,z,t) = w_0(x,y,t) \tag{1}$$

where

$$\xi = z\left(1 - \frac{4z^2}{3h^2}\right); \quad \xi^* = \frac{\partial \xi}{\partial z} = \left(1 - \frac{4z^2}{h^2}\right). \tag{2}$$

The strain displacement relations are written, using the above displacement forms, as

$$\varepsilon_x = \frac{\partial u_0}{\partial x} + \xi\frac{\partial u_1}{\partial x} - z\frac{\partial^2 w_0}{\partial x^2} + \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2$$

$$\varepsilon_y = \frac{\partial v_0}{\partial y} + \xi\frac{\partial v_1}{\partial y} - z\frac{\partial^2 w_0}{\partial y^2} + \frac{1}{2}\left(\frac{\partial w_0}{\partial y}\right)^2$$

$$\gamma_{xy} = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} + \xi\left(\frac{\partial u_1}{\partial y} + \frac{\partial v_1}{\partial x}\right) - 2z\frac{\partial^2 w_0}{\partial x\,\partial y} + \left(\frac{\partial w_0}{\partial x}\right)\left(\frac{\partial w_0}{\partial y}\right)$$

$$\gamma_{xz} = \xi^* u_1;\quad \gamma_{yz} = \xi^* v_1. \tag{3}$$

The constitutive relations and the definitions for stress-resultants are written as

$$\sigma_x = C_{11}\varepsilon_x + C_{12}\varepsilon_y;\quad \sigma_y = C_{12}\varepsilon_x + C_{22}\varepsilon_y$$

$$\tau_{xy} = C_{66}\gamma_{xy};\quad \tau_{xz} = C_{44}\gamma_{xz};\quad \tau_{yz} = C_{55}\gamma_{yz} \tag{4}$$

$$\begin{bmatrix} N_x & M_x & M'_x \\ N_y & M_y & M'_y \\ N_{xy} & M_{xy} & M'_{xy} \end{bmatrix} = \int \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} (1, -z, \xi)\,\mathrm{d}z. \tag{5}$$

The equilibrium equations in terms of stress-resultants are written as (Bhimaraddi, 1987a)

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = \int \rho \frac{\partial^2 u}{\partial t^2}\,\mathrm{d}z$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = \int \rho \frac{\partial^2 v}{\partial t^2}\,\mathrm{d}z$$

$$\frac{\partial M'_x}{\partial x} + \frac{\partial M'_{xy}}{\partial y} - \int \tau_{xz}\xi^*\,\mathrm{d}z = \int \rho \frac{\partial^2 u}{\partial t^2}\,\xi\,\mathrm{d}z$$

$$\frac{\partial M'_{xy}}{\partial x} + \frac{\partial M'_y}{\partial y} - \int \tau_{yz}\xi^*\,\mathrm{d}z = \int \rho \frac{\partial^2 v}{\partial t^2}\,\xi\,\mathrm{d}z$$

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x\,\mathrm{d}y} + \frac{\partial^2 M_y}{\partial y} = q + \frac{\partial}{\partial x}\left[N_x\frac{\partial w_0}{\partial x} + N_{xy}\frac{\partial w_0}{\partial y}\right] + \frac{\partial}{\partial y}\left[N_y\frac{\partial w_0}{\partial y} + N_{xy}\frac{\partial w_0}{\partial x}\right]$$

$$+ \int \rho\left(\frac{\partial^2 w_0}{\partial t^2} - z\frac{\partial^3 u}{\partial x\,\partial t^2} - z\frac{\partial^3 v}{\partial y\,\partial t^2}\right)\mathrm{d}z. \tag{6}$$

Expressing stress-resultants (5) in terms of five generalized displacement parameters using eqns (1)–(4) and substituting the same in the equilibrium equations, eqns (6), one obtains five governing equations in terms of five displacement parameters of the problem under consideration. The stress-resultants and the equilibrium equations in terms of displacements are given in the Appendix. The boundary conditions along the edge of the plate require that, either one member of each of the following six pairs or six linearly independent combinations of them must be specified:

along x = constant

$$N_x, u_0;\quad N_{xy}, v_0;\quad M'_x, u_1;\quad M'_{xy}, v_1;\quad Q_x, w_0;\quad M_x, \frac{\partial w_0}{\partial x};$$

along y = constant

$$N_{xy}u_0;\quad N_y v_0;\quad M'_{xy}u_1;\quad M'_y v_1;\quad Q_y w_0;\quad M_y\frac{\partial w_0}{\partial y}.$$

The governing equations for the classical TPT can be obtained by using $\xi = 0$ or alternatively by ignoring the terms and equations associated with u_1, and v_1 displacement parameters. Whereas, the governing equations for the Mindlin-type CST can be obtained by using $\xi = z$ in conjunction with the shear correction factor. Note that CST in the present case is not exactly the Mindlin plate theory but is similar to it in respect to its assumption of constant shear strain variation across the thickness. In the present CST, unlike the Mindlin theory, rotations of the mid-plane are split into two parts one corresponding to the well-known flexural rotation and the other corresponding to the shear rotation.

SOLUTION OF EQUILIBRIUM EQUATIONS

The closed-form solution to the coupled non-linear equilibrium equations (eqns (A5a)–(A5e)) is difficult to obtain. Here an approximate solution using the Galerkin method is sought. As an example, a simply supported plate has been considered and the following single-mode solution has been assumed for transverse displacement w_0:

$$w_0(x, y, t) = f(t)\sin\alpha x\sin\beta y \quad (\alpha = \pi/a;\ \beta = \pi/b) \tag{7}$$

and the other displacement parameters can be written as follows:

$$\begin{aligned}
u_0 &= a_1 f\cos\alpha x\sin\beta y + a_2 f^2\sin 2\alpha x + a_3 f^2\sin 2\alpha x\cos 2\beta y\\
&\quad + (a_{13}N_x^* + a_{14}N_y^* + a_{15})x\\
v_0 &= a_4 f\sin\alpha x\cos\beta y + a_5 f^2\sin 2\beta y + a_6 f^2\cos 2\alpha x\sin 2\beta y\\
&\quad + (a_{16}N_x^* + a_{17}N_y^* + a_{18})y\\
u_1 &= a_7 f\cos\alpha x\sin\beta y + a_8 f^2\sin 2\alpha x + a_9 f^2\sin 2\alpha x\cos 2\beta y\\
v_1 &= a_{10} f\sin\alpha x\cos\beta y + a_{11} f^2\sin 2\beta y + a_{12} f^2\cos 2\alpha x\sin 2\beta y.
\end{aligned} \tag{8}$$

Equations (8) have been written by neglecting the inertia terms appearing in eqns (6a)–(6d). Quantities a_1–a_{12} can be obtained by substituting eqns (8) in eqns (A5a)–(A5d) and equating the coefficients of like trignometric terms and the quantities a_{13}–a_{18} can be obtained by using the following boundary conditions:

$$\frac{1}{b}\int_0^b N_x|_{x=0,a}\,dy = N_x^*;\quad \frac{1}{a}\int_0^a N_y|_{y=0,b}\,dx = N_y^*. \tag{9}$$

Here N_x^* and N_y^* are the in-plane forces (tension positive) and are assumed to be time independent. It can be verified that the selection of eqns (8) satisfies the zero in-plane shear condition along the four edges of the plate. Substituting eqns (7) and (8) in eqn (A5e) and applying the Galerkin method one obtains the following equation:

$$\alpha_1\frac{d^2 f}{dt^2} + (\alpha_2 + \alpha_5 N_x^* + \alpha_6 N_y^*)f + \alpha_8 f^2 + \alpha_{10} f^3 = q. \tag{10}$$

It may be noted here that in obtaining eqn (10) the inertia terms associated with u_0, v_0, u_1, and v_1 in eqn (A5e) have been neglected. Considering the proportional loading ($N_y^* = \lambda N_x^*$, λ = constant, $N_x^* = N$) eqn (10) may be written as

$$\frac{d^2f}{dt^2}+\Omega_L^2 f+\frac{\alpha_8}{\alpha_1}f^2+\frac{\alpha_{10}}{\alpha_1}f^3=q \tag{11}$$

where

$$\Omega_L^2=\frac{\alpha_2}{\alpha_1}\left(1+\frac{N}{N_c}\right);\quad N_c=\frac{\alpha_2}{\alpha_5+\lambda\alpha_6}. \tag{12}$$

Here N_c and Ω_L correspond to the bifurcation buckling load in static compression and the linear free vibration frequency of an in-plane loaded perfectly flat plate, respectively.

Following exactly the same steps as above and incorporating the influence of initial imperfections ($\bar{w}_0$) one can obtain the following equation which governs the non-linear flexural response of an imperfect plate with in-plane loading:

$$\alpha_1\frac{d^2f}{dt^2}+(\alpha_2+\alpha_3f_0+\alpha_4f_0^2+\alpha_5N_x^*+\alpha_6N_y^*)f+(\alpha_8+\alpha_9f_0)f^2+\alpha_{10}f^3$$
$$+(\alpha_5N_x^*+\alpha_6N_y^*)f_0=q. \tag{13}$$

Here it is assumed that the initial imperfections are of the type

$$\bar{w}_0=f_0\sin\alpha x\sin\beta y. \tag{14}$$

Using eqn (13) various problems of the flexural response of plates can be addressed. Among others, the large amplitude vibration of an imperfect plate ($N=0$) and small amplitude vibration of a preloaded plate in the vicinity of its static buckled position are considered in this paper.

LARGE AMPLITUDE VIBRATIONS OF IMPERFECT PLATE

Equation (13), in the absence of in-plane and transverse ($N=q=0$) loads, can be written as

$$\frac{d^2f}{dt^2}+\delta_1f+\delta_2f^2+\delta_3f^3=0. \tag{15}$$

The above equation is a second-order ODE with quadratic and cubic nonlinearities. The solution of this equation can be obtained by using the method of multiple scales (Bhimaraddi, 1987b). The solution can be written as

$$\begin{aligned}f(t)&=A\cos\Omega+\tfrac{1}{2}A^2[c_1\cos2\Omega+c_2]+\tfrac{1}{4}A^3c_3\cos3\Omega\\&+\tfrac{1}{8}A^4[c_4\cos4\Omega+c_5\cos2\Omega+c_6]+\tfrac{1}{16}A^5[c_7\cos5\Omega+c_8\cos3\Omega]\\&+\tfrac{1}{32}A^6(c_9\cos6\Omega+c_{10}\cos4\Omega+c_{11}\cos2\Omega+c_{12}]\\&+\tfrac{1}{64}A^7[c_{13}\cos7\Omega+c_{14}\cos5\Omega+c_{15}\cos3\Omega]\end{aligned} \tag{16}$$

where $\Omega=\Omega_{NL}t+\Theta$, A and Θ are to be determined from the initial conditions and Ω_{NL} is the non-linear frequency which is dependent on the amplitude A in the following manner:

$$\Omega_{NL}=\sqrt{\delta_1}(1+b_1A^2+b_2A^4+b_3A^6) \tag{17}$$

and the expressions for b_1, b_2, and b_3 are given in a previous paper (Bhimaraddi, 1987b). Typical material properties considered in the computation of numerical results for a square composite (two-layered, 0/90) plate correspond to the following:

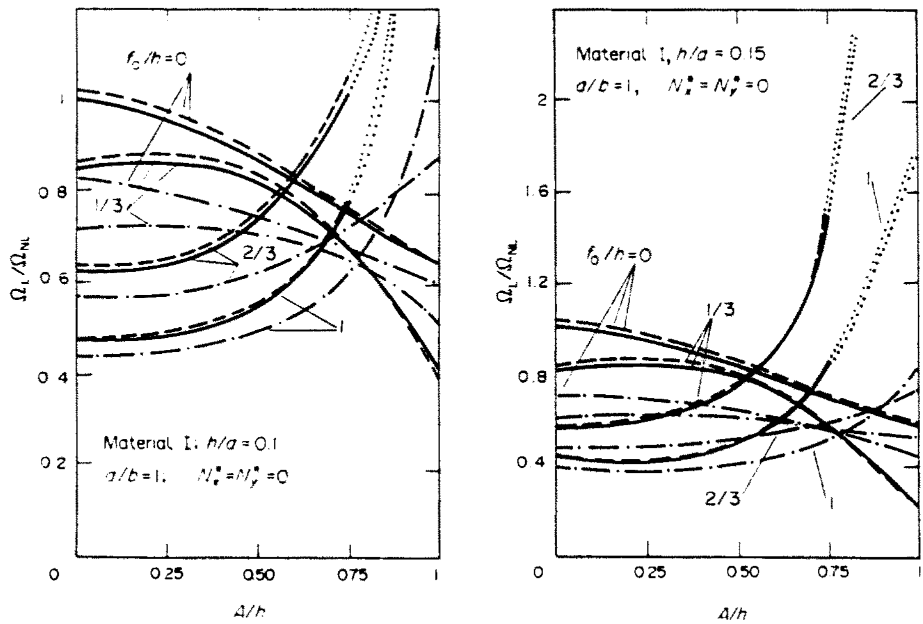

Fig. 1. Relation between non-linear frequency and amplitude for two-layered (0/90) square plate (——, PST; - - -, CST; — · — · —, TPT).

$$\frac{E_x}{E_y} = 40;\quad \frac{E_x}{G_{xy}} = \frac{E_x}{G_{xz}} = \frac{E_x}{G_{yz}} = 80;\quad \nu_{xy} = \tfrac{1}{4};\quad \rho_1 = \rho_2 = \rho \quad \text{(Material I)}$$

$$\frac{E_x}{E_y} = 10;\quad \frac{E_x}{G_{xy}} = \frac{E_x}{G_{xz}} = \frac{E_x}{G_{yz}} = 20;\quad \nu_{xy} = \tfrac{1}{4};\quad \rho_1 = \rho_2 = \rho \quad \text{(Material II)}.$$

To facilitate the comparison of various theories, values of Ω_L in (Ω_L/Ω_{NL}), (Ω_p/Ω_L) and N_c in (N/N_c) correspond to those obtained by using PST. The accuracy and the correctness of the algebra involved in the present work has been verified by comparing the solutions with those available in the literature for various cases such as: linear static and free vibration solutions (Bhimaraddi and Stevens, 1984), non-linear free vibration of isotropic plates solutions (Bolotin, 1964; Ambartsumian *et al.*, 1966), and non-linear vibrations of in-plane loaded plates (Bhimaraddi, 1987b). The shear correction factors used (in the case of CST) correspond to $\pi^2/12$.

Figure 1 shows the amplitude vs frequency variation of a two-layered (0/90) square plate with different initial imperfections. It is clearly seen that there is hardly any difference in the predictions of PST and CST. But there is a considerable difference between the shear deformable theories and the TPT.

As observed by Hui (1984) one can note that for lower imperfection values ($f_0/h \leqslant 1/3$) the response is of hardening type whereas, for higher imperfection values ($f_0/h > 1/3$) the response is of soft-spring type. It has been observed that for highly imperfect plates ($f_0/h \geqslant 2/3$), with amplitude vibration greater than $0.75h$, the perturbation method did not give meaningful results for shear deformable theories; and whereas, such an observation was not made in the case of TPT.

It may be seen that if the response is of the hardening type the difference between PST and TPT gradually diminish as the amplitude of vibration increases; whereas, if the response is of soft-spring type the opposite is true. This observation suggests that for hard-spring type response the shear deformation effects diminish as the amplitude of vibration increases and for soft-spring type response the shear deformation effects dominate with the increasing amplitude of vibration. Also comparing Figs 1(a) and (b) one can note that the greater the thickness of the plate the greater is the dominance of the shear deformation effects on the frequency.

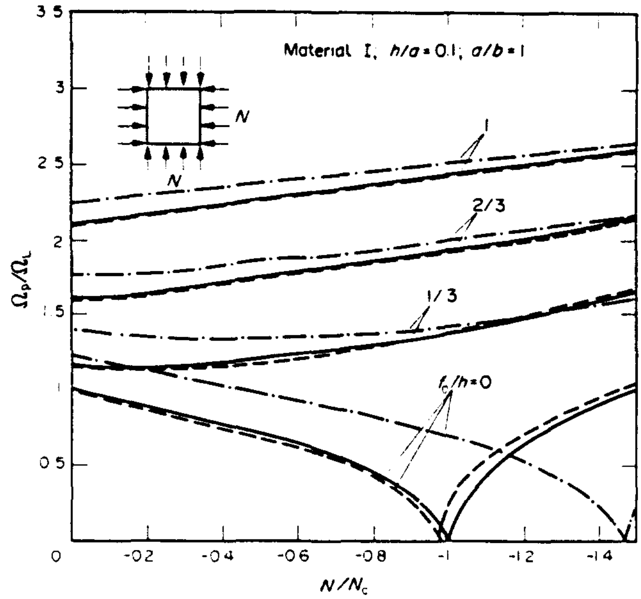

Fig. 2. Relation between frequency and the applied in-plane load for two-layered square plate.

SMALL AMPLITUDE VIBRATIONS OF BUCKLED PLATES

For this case the static part of eqn (13) is first considered in the absence of transverse load ($q = 0$) which may be written as

$$\delta_1 f_b + \delta_2 f_b^2 + \delta_3 f_b^3 + \frac{N}{N_c}(\delta_4 f_b + \delta_5) = 0. \quad (18)$$

Thus, for a given level of loading (N/N_c) and for a given value of initial imperfection one can solve eqn (18) to obtain the transverse displacement (f_b) prior to imposing free vibrations in the form of the buckling mode. Then at any instant of time the total displacement of the plate is given by

$$f_T = f_b + f_t. \quad (19)$$

Here f_t is the time-dependent part of the displacement. Now the equation of motion for the small amplitude vibration of a plate in the vicinity of its buckled position can be written, after substituting eqn (19) in eqn (13) and neglecting non-linear terms in f_t, as

$$\alpha_1 \frac{d^2 f_t}{dt^2} + \left(\delta_1 + 2\delta_2 f_b + 3\delta_3 f_b^2 + \frac{N}{N_c}\delta_4\right) f_t = 0. \quad (20)$$

In the above equation f_b has to be obtained from eqn (18). Before presenting the numerical results it can be shown that the term in parentheses in eqn (20) becomes zero when the load reaches the limit load. To do this consider again eqn (18) and rewrite it in the following manner:

$$\frac{N}{N_c} = -\left[\frac{\delta_1 f_b + \delta_2 f_b^2 + \delta_3 f_b^3}{\delta_4 f_b + \delta_5}\right]; \quad (21)$$

at limit load one has $(d/df_b)(N/N_c) = 0$, which leads to

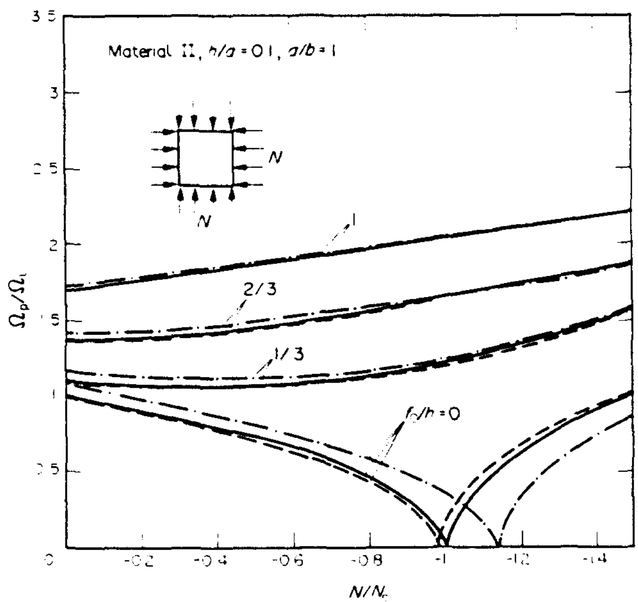

Fig. 3. Relation between frequency and the applied in-plane load for two-layered square plate.

$$2\delta_3\delta_4 f_L^3 + (\delta_2\delta_4 + 3\delta_3\delta_5) f_L^2 + 2\delta_2\delta_5 f_L + \delta_1\delta_5 = 0. \qquad (22)$$

Here f_L indicates the displacement at limit load. Substituting f_L in place of f_0, and N_L/N_c in place of N/N_c, via eqn (21), one can see that the term in parentheses in eqn (20) reduces to the left-hand side of eqn (22). Thus if the plate is imperfection sensitive, for which limit load is less than the critical load ($N_L < N_c$), the frequency goes on decreasing as the load level increases and becomes zero at limit load. However, for imperfection insensitive plates the frequency increases as the load increases.

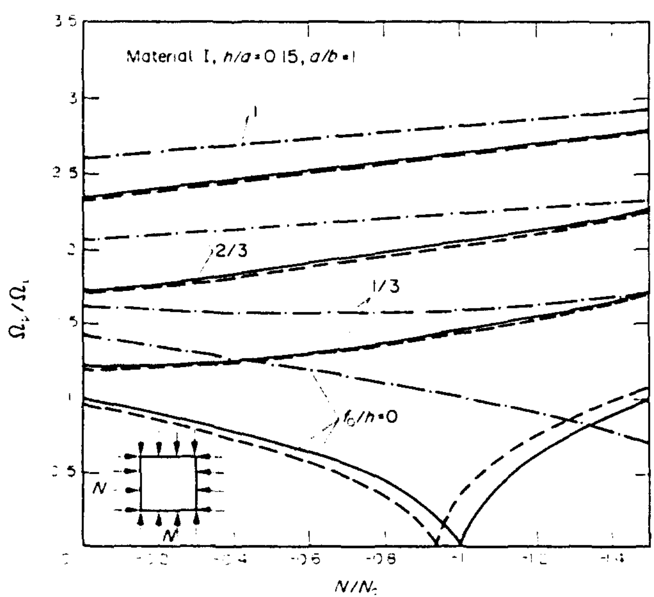

Fig. 4. Relation between frequency and the applied in-plane load for two-layered square plate.

Figures 2–4 illustrate the relation between the frequency of vibration and the preload for a square plate with two different material and overall thicknesses. It may be seen from these figures that the frequency becomes zero when the preload reaches the critical load for perfect plates. Also it may be noted that the frequency increases as the preload increases for imperfect plates. This suggests, as stated earlier, that the plate examples considered here are imperfection insensitive.

For perfect plates, it may be observed that CST under-estimates the frequency whereas TPT over-estimates the same when the preload is less than the critical load when compared with PST. For imperfect plates CST always under-estimates the frequency and TPT always over-estimates the same. Again one can note that the difference between CST and PST is hardly noticeable and that between TPT and PST is considerable. However, if the preload is high the difference between PST and TPT is very small. Comparing Figs 2 and 3 (also Figs 2 and 4), it may be said that the effects of shear deformation become increasingly evident as the plate thickness increases and as the material of the plate becomes highly anisotropic.

CONCLUSIONS

The vibration analysis of laminated composite plates including the shear deformation effects and the additional complicating effects such as: in-plane loading, initial imperfections, and the geometric nonlinearities has rarely been addressed in the literature. In this paper the non-linear response of imperfect composite plates has been considered using an accurate shear deformation theory. The difference between the predictions of the classical TPT and the shear deformation theory is considerable and the magnitude of this difference depends on the in-plane loading and the initial imperfections of the plate. For some combination of these parameters TPT may give as accurate results as those of shear deformation theories.

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APPENDIX

Force–displacement relations

$$\begin{bmatrix} N_x \\ N_y \\ M'_x \\ M'_y \\ M_x \\ M_y \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & \bar{B}_{11} & \bar{B}_{12} & B_{11} & B_{12} \\ & A_{22} & \bar{B}_{12} & \bar{B}_{22} & B_{12} & B_{22} \\ & & \bar{\bar{D}}_{11} & \bar{\bar{D}}_{12} & \bar{D}_{11} & \bar{D}_{12} \\ & & & \bar{\bar{D}}_{22} & \bar{D}_{12} & \bar{D}_{22} \\ & & & & -D_{11} & -D_{12} \\ & \text{Symm.} & & & & D_{22} \end{bmatrix} \begin{bmatrix} u_{0,x} + \frac{1}{2}w_{0,x}^2 \\ v_{0,y} + \frac{1}{2}w_{0,y}^2 \\ u_{1,x} \\ v_{1,y} \\ -w_{0,xx} \\ -w_{0,yy} \end{bmatrix} \tag{A1}$$

$$\begin{bmatrix} N_{xy} \\ M'_{xy} \\ M_{xy} \end{bmatrix} = \begin{bmatrix} A_{66} & \bar{B}_{66} & B_{66} \\ \bar{B}_{66} & \bar{\bar{D}}_{66} & \bar{D}_{66} \\ B_{66} & \bar{D}_{66} & -D_{66} \end{bmatrix} \begin{bmatrix} u_{0,y} + v_{0,x} + w_{0,x}w_{0,y} \\ u_{1,y} + v_{1,x} \\ -2w_{0,xy} \end{bmatrix} \tag{A2}$$

$$\begin{aligned} Q_x &= \frac{\partial M_x}{\partial x} + 2\frac{\partial M_{xy}}{\partial y} - N_x \frac{\partial w_0}{\partial x} - N_{xy}\frac{\partial w_0}{\partial y} + P_2 \frac{\partial^2 u_0}{\partial t^2} + P_5 \frac{\partial^2 u_1}{\partial t^2} - P_4 \frac{\partial^3 w_0}{\partial x \partial t^2} \\ Q_y &= \frac{\partial M_y}{\partial y} + 2\frac{\partial M_{xy}}{\partial x} - N_y \frac{\partial w_0}{\partial y} - N_{xy}\frac{\partial w_0}{\partial x} + P_2 \frac{\partial^2 v_0}{\partial t^2} + P_5 \frac{\partial^2 v_1}{\partial t^2} - P_4 \frac{\partial^3 w_0}{\partial y \partial t^2} \end{aligned} \tag{A3}$$

and

$$\begin{aligned} (A_{ij}, \bar{A}_{ij}, B_{ij}, \bar{B}_{ij}, D_{ij}, \bar{D}_{ij}, \bar{\bar{D}}_{ij}) &= \int C_{ij}(1, \xi^{*2}, z, \xi, z^2, z\xi, \xi^2)\,\mathrm{d}z \\ (P_1, P_2, P_3, P_4, P_5, P_6) &= \int \rho(1, z, \xi, z^2, z\xi, \xi^2)\,\mathrm{d}z. \end{aligned} \tag{A4}$$

Equilibrium equations in terms of displacements

$$\begin{aligned} & A_{11}\frac{\partial^2 u_0}{\partial x^2} + A_{66}\frac{\partial^2 u_0}{\partial y^2} + (A_{12} + A_{66})\frac{\partial^2 v_0}{\partial x \partial y} + \bar{B}_{11}\frac{\partial^2 u_1}{\partial x^2} + \bar{B}_{66}\frac{\partial^2 u_1}{\partial y^2} \\ & \quad + (\bar{B}_{12} + \bar{B}_{66})\frac{\partial^2 v_1}{\partial x \partial y} - B_{11}\frac{\partial^3 w_0}{\partial x^3} - (B_{12} + 2B_{66})\frac{\partial^3 w_0}{\partial x \partial y^2} + A_{11}\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial x^2} \\ & \quad + (A_{12} + A_{66})\frac{\partial w_0}{\partial y}\frac{\partial^2 w_0}{\partial x \partial y} + A_{66}\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial y^2} = P_1\frac{\partial^2 u_0}{\partial t^2} + P_3\frac{\partial^2 u_1}{\partial t^2} - P_2\frac{\partial^3 w_0}{\partial x \partial t^2} \end{aligned} \tag{A5a}$$

$$\begin{aligned} & (A_{12} + A_{66})\frac{\partial^2 u_0}{\partial x \partial y} + A_{22}\frac{\partial^2 v_0}{\partial y^2} + A_{66}\frac{\partial^2 v_0}{\partial x^2} + (\bar{B}_{12} + \bar{B}_{66})\frac{\partial^2 u_1}{\partial x \partial y} + \bar{B}_{22}\frac{\partial^2 v_1}{\partial y^2} \\ & \quad + \bar{B}_{66}\frac{\partial^2 v_1}{\partial x^2} - B_{22}\frac{\partial^3 w_0}{\partial y^3} - (B_{12} + 2B_{66})\frac{\partial^3 w_0}{\partial x^2 \partial y} + A_{22}\frac{\partial w_0}{\partial y}\frac{\partial^2 w_0}{\partial y^2} + A_{66}\frac{\partial w_0}{\partial y}\frac{\partial^2 w_0}{\partial x^2} \\ & \quad + (A_{12} + A_{66})\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial x \partial y} = P_1\frac{\partial^2 v_0}{\partial t^2} + P_3\frac{\partial^2 v_1}{\partial t^2} - P_2\frac{\partial^3 w_0}{\partial y \partial t^2} \end{aligned} \tag{A5b}$$

$$\begin{aligned} & \bar{B}_{11}\frac{\partial^2 u_0}{\partial x^2} + \bar{B}_{66}\frac{\partial^2 u_0}{\partial y^2} + (\bar{B}_{12} + \bar{B}_{66})\frac{\partial^2 v_0}{\partial x \partial y} + \bar{\bar{D}}_{11}\frac{\partial^2 u_1}{\partial x^2} + \bar{\bar{D}}_{66}\frac{\partial^2 u_1}{\partial y^2} - \bar{A}_{44}u_1 \\ & \quad + (\bar{\bar{D}}_{12} + \bar{\bar{D}}_{66})\frac{\partial^2 v_1}{\partial x \partial y} - \bar{D}_{11}\frac{\partial^3 w_0}{\partial x^3} - (\bar{D}_{12} + 2\bar{D}_{66})\frac{\partial^3 w_0}{\partial x \partial y^2} + \bar{B}_{11}\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial x^2} \\ & \quad + (\bar{B}_{12} + \bar{B}_{66})\frac{\partial w_0}{\partial y}\frac{\partial^2 w_0}{\partial x \partial y} + \bar{B}_{66}\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial y^2} = P_3\frac{\partial^2 u_0}{\partial t^2} + P_6\frac{\partial^2 u_1}{\partial t^2} - P_5\frac{\partial^3 w_0}{\partial x \partial t^2} \end{aligned} \tag{A5c}$$

$$\begin{aligned} & (\bar{B}_{12} + \bar{B}_{66})\frac{\partial^2 u_0}{\partial x \partial y} + \bar{B}_{22}\frac{\partial^2 v_0}{\partial y^2} + \bar{B}_{66}\frac{\partial^2 v_0}{\partial x^2} + (\bar{\bar{D}}_{12} + \bar{\bar{D}}_{66})\frac{\partial^2 u_1}{\partial x \partial y} + \bar{\bar{D}}_{22}\frac{\partial^2 v_1}{\partial y^2} \\ & \quad + \bar{\bar{D}}_{66}\frac{\partial^2 v_1}{\partial x^2} - \bar{A}_{55}v_1 - \bar{D}_{22}\frac{\partial^3 w_0}{\partial y^3} - (\bar{D}_{12} + 2\bar{D}_{66})\frac{\partial^3 w_0}{\partial x^2 \partial y} + \bar{B}_{22}\frac{\partial w_0}{\partial y}\frac{\partial^2 w_0}{\partial y^2} \\ & \quad + (\bar{B}_{12} + \bar{B}_{66})\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial x \partial y} + \bar{B}_{66}\frac{\partial w_0}{\partial y}\frac{\partial^2 w_0}{\partial x^2} = P_3\frac{\partial^2 v_0}{\partial t^2} + P_6\frac{\partial^2 v_1}{\partial t^2} - P_5\frac{\partial^3 w_0}{\partial y \partial t^2} \end{aligned} \tag{A5d}$$

$$
\begin{aligned}
&-B_{11}\frac{\partial^3 u_0}{\partial x^3}-(B_{12}+2B_{66})\left[\frac{\partial^3 u_0}{\partial x\,\partial y^2}+\frac{\partial^3 v_0}{\partial x^2\,\partial y}\right]-B_{22}\frac{\partial^3 v_0}{\partial y^3}-\bar{D}_{11}\frac{\partial^3 u_1}{\partial x^3}\\
&\quad-(\bar{D}_{12}+2\bar{D}_{66})\left[\frac{\partial^3 u_1}{\partial x\,\partial y^2}+\frac{\partial^3 v_1}{\partial x^2\,\partial y}\right]-\bar{D}_{22}\frac{\partial^3 v_1}{\partial y^3}+D_{11}\frac{\partial^4 w_0}{\partial x^4}+2(D_{12}+2D_{66})\frac{\partial^4 w_0}{\partial x^2\,\partial y^2}\\
&\quad+D_{22}\frac{\partial^4 w_0}{\partial y^4}-\left[B_{11}\frac{\partial^3 w_0}{\partial x^3}+(B_{12}+2B_{66})\frac{\partial^3 w_0}{\partial x\,\partial y^2}\right]\left(\frac{\partial w_0}{\partial x}\right)-2B_{66}\frac{\partial^2 w_0}{\partial x^2}\frac{\partial^2 w_0}{\partial y^2}\\
&\quad-\left[B_{22}\frac{\partial^3 w_0}{\partial y^3}+(B_{12}+2B_{66})\frac{\partial^3 w_0}{\partial x^2\,\partial y}\right]\left(\frac{\partial w_0}{\partial y}\right)-B_{11}\left(\frac{\partial^2 w_0}{\partial x^2}\right)^2-B_{22}\left(\frac{\partial^2 w_0}{\partial y^2}\right)^2\\
&\quad-2(B_{12}+B_{66})\left(\frac{\partial^2 w_0}{\partial x^2\,\partial y^2}\right)^2=q-P_2\frac{\partial^3 u_0}{\partial x\,\partial t^2}-P_2\frac{\partial^3 v_0}{\partial y\,\partial t^2}-P_5\frac{\partial^3 u_1}{\partial x\,\partial t^2}-P_5\frac{\partial^3 v_1}{\partial y\,\partial t^2}\\
&\quad+P_1\frac{\partial^2 w_0}{\partial t^2}-P_4\frac{\partial^4 w_0}{\partial x^2\,\partial t^2}-P_4\frac{\partial^4 w_0}{\partial y^2\,\partial t^2}+\frac{\partial}{\partial x}\left(N_x\frac{\partial w_0}{\partial x}\right)\\
&\quad+\frac{\partial}{\partial y}\left(N_y\frac{\partial w_0}{\partial y}\right)+\frac{\partial}{\partial x}\left(N_{xy}\frac{\partial w_0}{\partial y}\right)+\frac{\partial}{\partial y}\left(N_{xy}\frac{\partial w_0}{\partial x}\right).
\end{aligned}
\tag{A5e}
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